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Abelian Extensions of Quantum Logics

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1. INTRODUCTION

All known "quantum logics"—orthomodular lattices and posets, orthoalgebras, effect algebras, etc.—may be regarded as cancelative, unital partial abelian semigroups. [See Wilce (1995a, b) for pertinent definitions.] An abelian group may also be regarded as a cancelative, unital PAS—one in which every element is a unit. As noted in Wilce (1995b), every cancelative unital PAS L has a canonical ideal $A(L)$ which is an abelian group, and a canonical quotient by $A(L)$ which is an effect algebra. Thus, we may regard every cancellative, unital PAS as an *extension* of an effect algebra by an abelian group. The present paper begins a study of such extensions.

2. ABELIAN EXTENSIONS OF EFFECT ALGEBRAS

An effect algebra E is a cancelative PAS with a unique unit. An abelian group A is a cancelative PAS in which *every* element is a unit. Their Cartesian product, $E \times A$, is a cancelative, unital PAS under $(p, x) \oplus (q, y) := (p \oplus q, x + y)$ (defined if $p \perp q$) with units $1 + x, x \in A$.

Definition. The *abelian part* of a cancelative, unital PAS L is the set $A(L) := L^{\perp} = \{a \in L | \forall b \in L a \perp b\}$.

$A(L)$ is an abelian group. [*Proof*: $\forall a \in A(L)$, $\exists b \in L$ with $(1 \oplus a) \oplus b = 1$, whence $1 \oplus (a \oplus b) = 1$. Thus $a \oplus b = 0$, so $b \in A(L)$, where it functions as $-a$.] For all $a, b \in L$, say $a \approx b$ iff $b = a \oplus x$ for some $x \in$

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$A(L)$. It is not hard to show that $\approx$ is a congruence on L. The following is proved in (Wilce, 1995b):

Theorem 1. $L_+ := L/\simeq$ is an effect algebra, and the quotient map $\pi: L \to L_+$ is a faithful homomorphism.

We thus have (in an obvious sense) a short exact sequence

$$0 \to A(L) \xrightarrow{i} L \xrightarrow{\pi} L_+ \to 0$$

In other words, L is an *extension* of the effect algebra L_+ by the abelian group $A(L)$. We shall say that L *splits* iff the foregoing extension does, i.e., iff $L \simeq A(L) \times L_+$. The proof of the following is routine:

Lemma 1. The following are equivalent:
(a) $L \simeq L_+ \times A$.
(b) $\exists \sigma \in \text{Hom}(L_+, L)\ \pi(\sigma(\text{p})) = p\ \forall p \in L_+$.
(c) $\exists \mu \in \text{Hom}(L, A)\ \mu(x) = x\ \forall x \in A(L)$.

Corollary 1. If L_+ is either a finite Boolean algebra or a finite chain, then L splits.

Proof. Any partial section of π over the atoms of L_+ extends to a homomorphic section. ■

3. COHOMOLOGY

We would like to be able to classify, up to isomorphism, the possible extensions L of a given effect algebra E by a given abelian group A. This can be done by means of cocycles, exactly as in the theory of extensions of abelian groups. Let $\sigma: L_+ \to L$ be *any* section of π with $\sigma(0) = 0$. If $p,q \in L_+$ with $p \perp q$, let $\beta(p, q)$ be the unique $x \in A(L)$ with $\sigma(a) \oplus \sigma(b) \oplus x = \sigma(a \oplus b)$. Then β satisfies

(1) $\beta(0, 0) = 0$
(2) $\beta(p, q) = \beta(q, p)$
(3) $\beta(p, q) + \beta(p \oplus q, r) = \beta(p, q \oplus r) + \beta(q, r)$
for all jointly orthogonal p, q, r in L_+.

Definition. Let E be an effect algebra and A an abelian group. An A-valued *cocycle* on E is a map $\beta: \perp_E \to A$ satisfying conditions (1)–(3).

All possible extensions of E by A are encoded by A-valued cocycles, as follows: If β is an A-valued cocycle on an effect-algebra E, let $E \times_\beta A = (E \times A, \oplus_\beta)$, where $(p, x) \oplus_\beta (q, y) := (p \oplus q, x + y + \beta(p, q))$. Then $E \times_\beta A$ is a cancelative, unital PAS, $A(E \times_\beta A) \simeq A$, and $(E \times_\beta A) \simeq E$.

Moreover, if $E = L_+$ and $A = A(L)$ for some cancelative, unital PAS L and arises from a section σ: $E \to L$, as discussed above, then $L \simeq E \times_\beta A$ via the map $a \mapsto (\pi(a), \sigma(\pi(a)))$.

It remains to determine when two cocycles give rise to isomorphic extensions. That they need not *always* do so is illustrated by the following

Example.[3] The *Diamond* is the four-element effect algebra $D = \{0, a, b, 1\}$ where $a = a'$ and $b = b'$. Consider the $\mathbb{Z}_2$-valued cocycle $\beta(a, a) = 0$, $\beta(b, b) = 1$. (Note that this *is* a cocycle, vacuously.) The extension $D \times_\beta \mathbb{Z}_2$ is not isomorphic to $D \times \mathbb{Z}_2$. An easy way to see this is to note that, in the former, the elements (1, 0) and (1, 1) each have two "half-elements" [$(a, 0)$ and $(a, 1)$ and $(b, 0)$, $(b, 1)$, respectively] while in the latter, (1, 0) has four half-elements and (1, 1) has none.

The set $Z^2(E, A)$ of all A-valued cocycles on L is an abelian group under pointwise addition, the zero element being the trivial cocycle $\beta(p, q) = 0$ corresponding to the trivial extension $E \times A$. Any function g: $E \to A$ with $g(0) = 0$ gives rise to a cocycle

$$\partial g(p, q) := g(p) + g(q) - g(p \oplus q)$$

The set $B^2(E, A)$ of such cocycles δg is a subgroup of $Z^2(L, A)$. The *second cohomology group* of E with coefficients in A is

$$\mathrm{Ext}(L, A) := Z^2(L, A)/B^2(L, A)$$

Exactly as in the context of abelian group extensions, one has:

Theorem 2. The cocycles β and β' in $Z^2(E, A)$ determine isomorphic extensions of E by A iff $\beta - \beta' \in B^2(E, A)$.

Thus, $\mathrm{Ext}(E, A)$ parametrizes the distinct isomorphism classes of extensions of E by A. Just for fun, let us compute $\mathrm{Ext}(E, A)$ in some easy cases:

Example. Here is another view of the Diamond example above: $Z^2(D, \mathbb{Z}_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ in an obvious way. If g: $D \to \mathbb{Z}_2$, then for $x = a, b$ we have

$$\partial(x, x) = g(1) - (g(x) + g(x)) = g(1)$$

Hence $\mathrm{Ext}(D, \mathbb{Z}_2) \simeq \mathbb{Z}_2$.

In computing the extensions of an orthoalgebra by an abelian group, it is often expedient to make use of an associated *algebraic test space* (Foulis *et al.*, 1992).

Example. Let E be the logic of the "wright triangle," i.e., the algebraic test space

[3] With thanks to Don Hadwin.

$$\{\{a, x, b\}, \{b, y, c\}, \{c, z, a\}\}$$

(Thus, the points of the foregoing structure correspond to the atoms of E, and the blocks to sets of atoms summing to 1.) The cocycles on E with values in an abelian group A are determined by a free assignment of values in A to pairs of orthogonal atoms, plus one free assignment to, say, the pair (a', a). One finds that Ext $(E, A) = 0$ for any A.

Example. Let E be the logic of the algebraic test space

$$\{\{a, x, b\}, \{b, y, c\}, \{c, z, x\}, \{a, z, y\}\}$$

(sometimes called the *flying wedge*). Then $\mathrm{EXT}(E, \mathbb{Z}_2)$ is nontrivial (as can be seen via a dimension count). Note that this test space does not have a separating set of states.

4. ABELIAN HULLS

Foulis and Bennett (1995) have observed that any effect algebra is associated with a *universal abelian group.* The same construction is available more generally for partial abelian semigroups. By a *measure* on a PAS L, we mean a homomorphism $\mu: L \to K$, where K is an abelian group.

Definition. Let L be a PAS, $\mathbb{Z}^{[L]}$ the free abelian group generated by L, and J the subgroup generated by elements of the form $p + q - (p \oplus q)$, where $p \perp q$ in L. The *abelian hull* of L is the group $G(L) := \mathbb{Z}^{[L]}/J$.

Note that there is a canonical measure $\nu_L: L \to G(L)$ given by $\nu_L(a) = [\chi_a]$ ($[\cdot]$ the quotient map $\mathbb{Z}^L \to G(L)$ and χ_a the $\mathbb{Z}$-valued characteristic function of $\{a\}$.) The following is straightforward (L, M arbitrary PASs):

Theorem 3. For any $\phi \in \mathrm{Hom}(L, M)$, $\exists! G(\phi) \in \mathrm{Hom}(G(L), G(M))$ with $G(\phi) \circ \nu_L = \nu_M \circ \phi$.

Remark. Theorem 3 sets up an isomorphism between $\mathrm{Hom}(G(L), K)$ and the group $M(L, K)$ of K-valued measures on L. Taking $K = \mathbb{C}^1$, the circle group, we have $M(L, \mathbb{C}^1) \simeq G(L)$, the Pointryagin dual of $G(L)$. Thus, for finite L, we have $G(L) \simeq M(L, \mathbb{C}^1)$.

Suppose now that $A = A(L)$. Applying G to the exact sequence

$$0 \to A \overset{i}{\to} L \overset{\pi}{\to} L_+ \to 0$$

we obtain a sequence

$$0 \to A \overset{G(i)}{\to} G(L) \overset{G(\pi)}{\to} G(L_+) \to 0$$

This may fail to be exact. Setting $B := \ker(G(\pi))$, we have the following commutative diagram, with short exact rows:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & L & \to & L_+ & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B & \to & G(L) & \to & G(L_+) & \to & 0 \end{array}$$

Lemma 2. The following are equivalent:
(a) L_+ hosts a separating set of measures.
(b) ν_+: $L_+ \to G(L_+)$ and $\nu|_A$: $A \to G(L)$ are injective.
(c) ν:$L \to G(L)$ is injective.
(d) L hosts a separating set of measures.

Proof. Obviously (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) $\Rightarrow$ (a). The link (b) $\Rightarrow$ (c) is supplied by the short five lemma (which is valid in this context). ■

Theorem 4. Let L be a PAS such that $\nu|_{A(L)}$: $A(L) \to G(L)$ is injective. The following are equivalent:
(a) $L \simeq A(L) \times L_+$.
(b) $G(L) \simeq A(L) \times G(L_+)$.
(c) $G(L) \simeq B \times G(L_+)$.

Proof. It is easily checked that $G(A \times L_+) \simeq A \times G(L_+)$. Thus, (a) entails (b). Obviously, (b) entails (c) [since it entails that $A(L) = B$.] Now suppose (c) holds, i.e., that $G(L)$ splits as $B \times G(L_+)$. Then we have a B-valued measure μ on L such that $\mu(b) = b$ for each $b \in B$. But then for every $a \in A$, we have $\mu(\nu(a)) = \nu(a)$, where ν: $L \to G(L)$ is the canonical map. So long as $\nu|_A$ is injective, therefore, $A \to L \to L_+$ also splits. ■

Any extension of a torsion-free abelian group by an abelian group of finite order is split (Rotman, 1995). Hence, by Theorem 4, we have:

Corollary 2. If L hosts a separating set of measures, $G(L_+)$ is torsion-free, and $A(L)$ has finite order, then $L \simeq A(L) \times L_+$.

Note that if $G(L_+)$ is torsion-free, L_+ contains no copy of D: If $a, b \in L_+$ with $a \oplus a = b \oplus b$, then $2(a - b) = 0$ in $G(L)$.

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